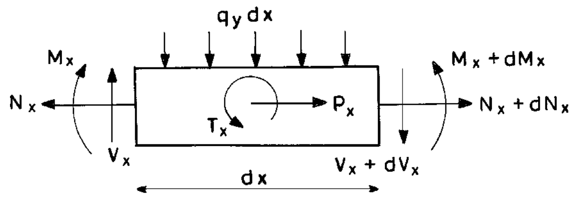

Fig. A1 Beam element subjected to different loads.

The governing equation for the beam subjected to lateral loading and distributed moment can be obtained by substituting Eq. (A2) in Eq. (A1),

$$-EI\frac{\partial^4 v}{\partial x^4} + \frac{\partial T_x}{\partial x} + q_y = 0 \tag{A3}$$

In-plane displacement due to bending of the beam

$$u^b = -y\frac{\partial v}{\partial x} \tag{A4}$$

Equilibrium of in-plane forces gives

$$\frac{\partial N_x}{\partial x} + p_x = 0 \tag{A5}$$

where p_x is distributed in-plane load.

For a beam of width b and thickness t,

$$N_x = bt\sigma_x, \qquad \sigma_x = E\epsilon_x = E\frac{\partial u_a}{\partial x} \tag{A6}$$

where u^a is in-plane displacement due to axial load. Sustitute Eq. (A6) in Eq. (A5) and integrate:

$$u^a = \frac{1}{btE}\int_x\int_x -p_x \,\mathrm{d}x\,\mathrm{d}x \tag{A7}$$

The total in-plane displacement due to bending and axial force is

$$u = u^b + u^a = -y\frac{\partial v}{\partial y} + \frac{1}{btE}\int_x\int_x -p_x \tag{A8}$$

References

[1]Gordaninejad, F., and Bert, C., W., "A New Theory for Bending of Thick Sandwich Beams," *International Journal of Mechanical Sciences*, Vol. 31, No. 11/12, 1989, pp. 925–934.

[2]Frostig, Y., Baruch, M., Vilnay, O., and Sheinman, I., "High-Order Theory for Sandwich-Beam Behavior with Transversely Flexible Core," *Journal of Engineering Mechanics*, Vol. 118, No. 5, 1992, pp. 1026–1043.

[3]Swanson, S. R., "An Examination of a Higher Order Theory for Sandwich Beams," *Composite Structures*, Vol. 44, 1999, pp. 169–177.

[4]Pai, F. P., and Palazotto, A. N., "A Higher-order Sandwich Theory Accounting for 3-D Stresses," *International Journal of Solids and Structures*, Vol. 38, 2001, pp. 5045–5062.

[5]Perel, V. Y., and Palazotto, A. N., "Finite Element Formulation for Cylindrical Bending of a Transversely Compressible Sandwich Plate, Based on Transverse Strains," *International Journal of Solids and Structures*, Vol. 38, 2001, pp. 5373–5409.

[6]Vinson, J. R., "Sandwich Structures," *Applied Mechanics Review*, Vol. 54, No. 3, 2001, pp. 201–214.

[7]Timoshenko, S. P., and Goodier, J. N., *Theory of Elasticity*, McGraw–Hill, New York, 1982.

[8]Arya, H., Shimpi, R. P., and Naik, N. K., "A Zigzag Model for Laminated Composite Beams," *Composite Structures*, Vol. 56, 2002, pp. 21–24.

[9]Pagano, N. J., "Exact Solutions for Composite Laminates in Cylindrical Bending," *Journal of Composite Materials*, Vol. 3, 1969, pp. 398–411.

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Optimal Fiber Angles to Resist the Brazier Effect in Orthotropic Tubes

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Introduction

As a thin-walled, circular, cylindrical shell is subjected to bending deflections, it will tend to ovalize according to the Brazier[1] effect. In doing so, the diminishing cross-sectional second moment of area of the tube reduces the flexural stiffness of the structure. This Technical Note details the ply configurations of composite tubes that maximize the critical failure (Brazier[1]) moment, where the reduced second moment of area is no longer able to sustain the applied moment.

Vlasov's[2] semimembrane theory assumes that longitudinal bending of a tube is resisted principally by A_{11} (longitudinal stiffness) terms and cross-sectional deformation by D_{22} (circumferential bending stiffness). This dependency is confirmed both by Kedward,[3] who expanded Brazier[1] analysis to include orthotropic shells, as well as by Harursampath and Hodges.[4] They found the critical failure load of a thin-walled cylinder under bending to be

$$M_{\mathrm{cr}} = 3.42a\sqrt{A_{11}\cdot D_{22}} \tag{1}$$

$$M_{\mathrm{cr}} = 4.223a\sqrt{A_{11}\cdot D_{22}} \tag{2}$$

respectively (where a is the tube radius). Despite coefficient differences, both models confirm that a maximization of limit moment must maximize $A_{11}\cdot D_{22}$.

This Technical Note provides an analytical optimization of shell composition with respect to the critical Brazier[1] moment, which will be useful in the design of structures where material failure and local buckling are not likely to occur.

Ply Configuration

The A_{11} term is a tensile/compressive stiffness and, thus, dependent solely on cross-sectional area of the plies. As such, it is not a function of stacking order, but only of the average longitudinal stiffness of plies, and is maximized by including as many 0-deg plies as possible. Conversely, the transverse shell bending stiffness D_{22} is stacking order dependent, being a function of the shell wall second moment of area. The maximization of this term is dependent on the placement of circumferential (90-deg) plies far from the shell wall's neutral axis.

It, therefore, appears intuitive that in highly unidirectional composites, the optimal layup will consist of 90-deg plies sandwiching a 0-deg layer. To test this hypothesis, a generalized orthotropic, symmetric three-layer configuration of balanced ($\pm\theta$) plies, $[\pm\theta(O), \pm\theta(M)]_s$ (Fig. 1) is studied to determine the optimum ply angle of each layer. The validity of only three layers will be analyzed subsequently, and the optimum relative layer thicknesses will be found.

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Analysis

From standard material section property definitions,

$$A_{11} = \sum_{k=1}^{N} (\bar{Q}_{11})_k \times (z_k - z_{k-1}) \tag{3}$$

$$D_{22} = \frac{1}{3}\sum_{k=1}^{N} (\bar{Q}_{22})_k \times \left(z_k^3 - z_{k-1}^3\right) \tag{4}$$

where z is the ply distance to the shell neutral axis and $[\bar{Q}]$ is the transformed reduced stiffness matrix. With three layers arranged as in Fig. 1, Eqs. (3) and (4) become

$$A_{11} = (\bar{Q}_{11}t)_M + (\bar{Q}_{11}t)_O$$

$$D_{22} = (\bar{Q}_{22}I)_M + (\bar{Q}_{22}I)_O$$

where t is the layer thickness and I is the relevant second moment of area of the layer about the shell wall's neutral axis. This gives an objective function to be maximized of

$$A_{11} \cdot D_{22} = [(\bar{Q}_{11}t)_M + (\bar{Q}_{11}t)_O] \cdot [(\bar{Q}_{22}I)_M + (\bar{Q}_{22}I)_O] \tag{5}$$

which may be written in terms of ply angle by using invariant material relationships.[5] To find the optimum ply orientations, an equal overall thickness of middle and outer plies was initially used, and the total $A_{11} \cdot D_{22}$ calculated for various inner and outer layer ply angles. This is plotted in Fig. 2 as a critical Brazier[1] moment [Eq. (1)] for a typical E-glass fiber-reinforced plastic (GFRP) with an E_{11}:E_{22}:G_{12} ratio of 10:1:2 and $\upsilon_{12} = 0.3$.

This has been nondimensionalized by dividing by the corresponding value for a quasi-isotropic layup, a homogeneous layup consisting of equal numbers of 0-, 90-, −45- and 45-deg plies that serves as a useful reference layup. In Fig. 2, the darker areas represent improved performance, with the darkest area representing a 39% increase in limit moment over the quasi-isotropic layup and a 92% increase over the weakest design. The dashed line shows the performance of a quasi-isotropic layup (relative performance of 1.0), and along the dotted line the outer and middle layers are at the same angle, effectively generating a single layer. It is apparent from Fig. 2 that, for this example, the optimum configuration is obtained with outer 90-deg plies sandwiching a 0-deg middle layer, as was initially hypothesized. The plot is topologically similar for any composite where $E_{11} > E_{22}$ and $G_{12} < G_{\text{iso}}$ (shear stiffness of the quasi-isotropic layup), and, therefore, any tube made with such a composite will be optimized by this configuration.

If $G_{12} > G_{\text{iso}}$, then the high shear stiffness means that 45-deg plies have a significant stiffening effect, such that configurations consisting of nonorthogonal plies will be optimal. However, composites with such properties are not common, and they will not be discussed here.

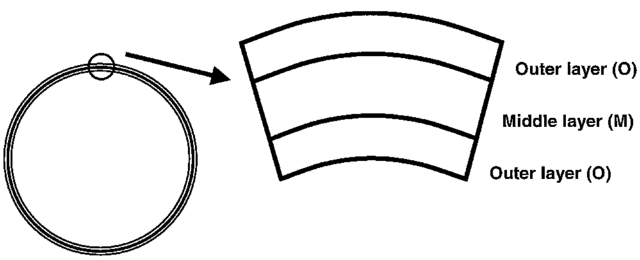

Fig. 1 Layer arrangement and nomenclature.

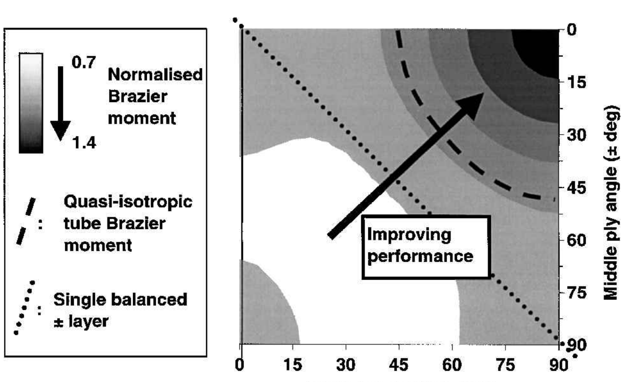

Fig. 2 Variation of Brazier[1] moment $(A_{11}D_{22})^{1/2}$ with ply angle for three-layer GFRP tube.

Optimization of Ply Thickness

If the ply orientations are $\theta = 0$ and 90 deg, then

$$\bar{Q}_{11}(\text{0-deg plies}) = \bar{Q}_{22}(\text{90-deg plies}) = Q_{11}$$

$$\bar{Q}_{22}(\text{0-deg plies}) = \bar{Q}_{11}(\text{90-deg plies}) = Q_{22}$$

Thus, Eq. (5) becomes

$$A_{11} \cdot D_{22} = \left(t_{\text{tot}}^4/12\right) \cdot \left[-(Q_{11} - Q_{22})^2 \cdot \left(t_M^4/t_{\text{tot}}^4\right) - (Q_{11} - Q_{22}) \times Q_{22} \cdot \left(t_M^3/t_{\text{tot}}^3\right) + (Q_{11} - Q_{22})Q_{11} \cdot (t_M/t_{\text{tot}}) + Q_{11} \cdot Q_{22}\right] \tag{6}$$

where t_M/t_{tot} is the proportion of 0-deg plies to the overall thickness. When Eq. (6) is differentiated with respect to t_M, equated to zero (to find a maximum) and substituted for,

$$Q_{11} = E_{11}/(1 - \upsilon_{12}\upsilon_{21}), \qquad Q_{22} = E_{22}/(1 - \upsilon_{12}\upsilon_{21})$$

where E_{ij} is the appropriate Young's modulus and ν_{ij} the Poisson's ratio, which gives

$$\frac{\partial(A_{11} \cdot D_{22})}{\partial(t_M)} = \frac{t_{\text{tot}}^3}{12}\left[\frac{E_{11} - E_{22}}{(1 - \upsilon_{12}\upsilon_{21})^2}\right]\left[-4(E_{11} - E_{22}) \cdot \frac{t_M^3}{t_{\text{tot}}^3} - 3E_{22} \cdot \frac{t_M^2}{t_{\text{tot}}^2} + E_{11}\right] = 0 \tag{7}$$

This cubic expression is readily solved and plotted for various ratios of E_{11} to E_{22} because the optimum is purely a function of this ratio and not of Poisson's ratio or shear modulus (if $G_{12} < G_{\text{iso}}$). The variation of optimum ply ratio with E_{22}/E_{11} is fairly linear and can be approximated by

$$t_M/t_{\text{tot}} = 0.63 - 0.05(E_{22}/E_{11}) \tag{8}$$

For most polymeric matrix composites with E_{22}/E_{11} ratios of 0.1–0.2 (0.2 for GFRP) the optimal value of t_M/t_{tot} is around 0.62. This value of t_M/t_{tot} approximately doubles $A_{11} \cdot D_{22}$ (for GFRP) compared to a quasi-isotropic layup. The collapse moment M_{cr} [Eqs. (1) and (2)] is, therefore, increased by 42%.

Note that this model assumes that plies can be subdivided indefinitely. For plies of discrete thickness, the nearest ratio should be selected.

Multiple Layers

The optimum position of the 90-deg plies is driven by the need to maximize D_{22}, and, as such, they are best placed farthest from the shell wall's neutral axis. The optimal location of the 0-deg plies follows suit and, by default, lies at the neutral axis. However, it may be that the presence of additional intermediate layers with different ply angles may increase the collapse moment. To model this, intermediate C layers have been introduced, with thickness t_C and ply

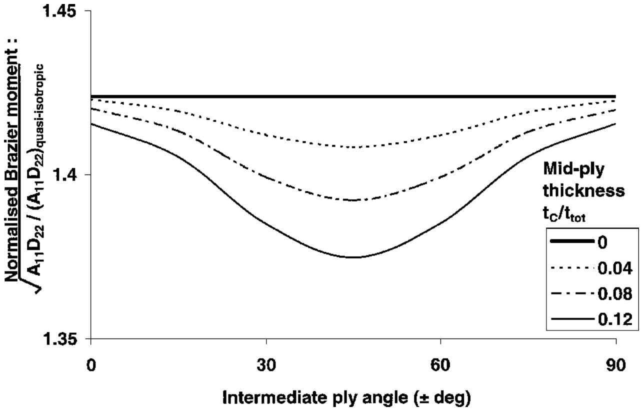

Fig. 3 Effect of inserting intermediate ply on Brazier[1] moment $(A_{11} \cdot D_{22})^{1/2}$ of a GFRP tube.

angles $\pm\theta_C$ (Fig. 3), between the middle and outer layers. These layers were introduced into Eq. (5), giving

$$A_{11} \cdot D_{22} = [(\bar{Q}_{11}t)_M + (\bar{Q}_{11}t)_O + (\bar{Q}_{11}t)_C] \times [(\bar{Q}_{22}I)_M + (\bar{Q}_{22}I)_O + (\bar{Q}_{22}I)_C]$$

as an expression for the Brazier[1] moment. Once again, the effect of varying ply angles and intermediate layer thickness (for a fixed overall shell thickness) can be found using the Tsai–Pagano[5] relationships. The effect on Brazier[1] moment of varying t_C and $\pm\theta_C$ is plotted in Fig. 3 using the optimum GFRP layup [90, 0-deg]$_s$ and t_M/t_{tot} of 0.6. As is apparent, the introduction of another, nonorthogonal, layer always reduces the Brazier[1] moment, suggesting that the optimal configuration consists solely of three layers.

Conclusions

The optimum configuration for a tube under bending is independent of Poisson's ratio and largely invariant with E_{11} and E_{22} for common highly directional composite materials. The optimal ply configuration for typical composite tubes is found to be 90–0–90 deg, with 62% 0-deg plies. Such a configuration increases the maximum bending moment in GFRP tubes by 42% compared to a quasi-isotropic configuration.

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References

[1]Brazier, L. G., "The Flexure of Thin Cylindrical Shells and Other 'Thin' Sections," *Late of the Royal Aircraft Establishment*, Repts. and Memoranda 1081 (M.49.), Farnborough, England, U.K., 1926, pp. 1–30.

[2]Vlasov, V. Z., "General Theory of Shells and Its Applications to Engineering," NASA Technical Translation TTF-99, 1964.

[3]Kedward, K. T., "Nonlinear Collapse of Thin-Walled Composite Cylinders Under Flexural Loading," *Proceedings of the 2nd International Conference on Composite Materials*, Toronto, 1978, pp. 353–365.

[4]Harursampath, D., and Hodges, D. H., "Asymptotic Analysis of the Non-Linear Behaviour of Long Anisotropic Tubes," *International Journal of Non-Linear Mechanics*, Vol. 34, No. 2, 1999, pp. 1003–1018.

[5]Tsai, S. W., and Pagano, N. J., "Invariant Properties of Composite Materials," *Composite Materials Workshop*, Technomic, Westport, CT, 1968, pp. 233–253.

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Thermal Postbuckling of Uniform Columns: A Simple Intuitive Method

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Nomenclature

A	=	area of cross section of the beam
a	=	central lateral deflection of the column
E	=	Young's modulus
I	=	area moment of inertia
L	=	length of the beam
P_t	=	thermal load
r	=	radius of gyration
T	=	temperature
U	=	strain energy as given by Eq. (1)
u	=	axial displacement
W	=	work done by the thermal load as given by Eq. (2)
w	=	lateral displacement
x	=	axial coordinate
α	=	coefficient of thermal expansion
β_1, β_2	=	coefficients in Eqs. (3) and (4)
ε_x	=	axial strain including nonlinear terms
λ_L	=	linear thermal critical load, $(\alpha T L^2/r^2)_L$
λ_{NL}	=	thermal post buckling load, $(\alpha T L^2/r^2)_{NL}$
λ_{Ta}	=	axial tension parameter as given by Eq. (5)
ψ_x	=	curvature

Introduction

THE truss- or frame-type structure is widely used in rocket and space structures, for example, interstages, solar sails, etc. These structures are exposed to thermal loading because of aerodynamic or solar heating. Structural elements such as uniform columns are the basic components of these structures, and the prediction of their postbuckling behavior is an important design input. Furthermore, their postbuckling strength can be effectively used in achieving an optimum (minimum mass) design of these structural elements.

Postbuckling behavior of columns subjected to mechanical loads was discussed by Dym[1] and Thompson and Hunt[2] using a differential equation approach. Although this approach gives exact solutions for some simple structural configurations, it is not easy to obtain solutions when complex geometries, loads, and boundary conditions are involved. As such, one has to resort to numerical methods to obtain solutions for such structural configurations. A numerical method, such as the versatile finite element method, and a continuum method, such as the Rayleigh–Ritz method, were used to investigate the thermal postbuckling behavior of uniform columns with immovable ends by Rao and Raju.[3] However, these methods are relatively tedious. Even though the finite element formulation is highly versatile, one has to idealize the column with a large number of elements, and the solution involves a large number of iterations, to achieve a desired degree of accuracy in evaluating λ_{NL}.

In the present Note, a simple, intuitive method is proposed to predict the thermal postbuckling behavior of uniform columns with different boundary conditions, after briefly describing the Rayleigh–Ritz formulation.

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